A connected component of the moduli space of surfaces of general type with $p_g = 0$

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Abstract

Let S be a minimal surface of general type with $p_g(S) = 0$ and such that the bicanonical map $\phi : S \to \mathbf{P}^{K_S^2}$ is a morphism: then the degree of ϕ is at most 4 by [7], and if it is equal to 4 then $K_S^2 \leq 6$ by [8]. We prove that if $K_S^2 = 6$ and $\deg \phi = 4$ then S is a so-called *Burniat surface* (see [12]). In addition we show that minimal surfaces with $p_g = 0$, $K^2 = 6$ and bicanonical map of degree 4 form a 4-dimensional irreducible connected component of the moduli space of surfaces of general type.

1 Introduction

Let S be a minimal surface of general type over the complex numbers with $p_g(S) = 0$, and denote by $\phi : S \to \mathbf{P}^{K_S^2}$ the bicanonical map: in [7], the first author has proven that if $K_S^2 \geq 5$, or if $K_S^2 = 3, 4$ and ϕ is a morphism, then the degree of ϕ is ≤ 4. This result is made more precise in [8], where it is proven that if $\deg \phi = 4$, then $K_S^2 \leq 6$. The latter bound is sharp, as it is shown by the the so-called *Burniat surfaces* (see [12] and [3]): these are surfaces of general type with $p_g = 0$, $2 \leq K^2 \leq 6$ whose bicanonical map is 4-to-1 onto a smooth Del Pezzo surface. Burniat surfaces arise as minimal desingularizations of $\mathbf{Z}_2 \times \mathbf{Z}_2$-covers of $\mathbf{P}^2$ branched on certain arrangements of lines; a direct construction for the case $K^2 = 6$ is given in section 3.

Here we concentrate on the "borderline case", namely $K_S^2 = 6$. We start by showing that *all* these surfaces have smooth bicanonical image and ample canonical class. This is an unexpected feature, and may perhaps be related

to the fact that, although these surfaces have moduli (see theorem 1.2), they are expected to be rigid.

The smoothness of the bicanonical image is the starting point for a very detailed analysis of the geometry of these surfaces that enables us to prove the following:

Theorem 1.1 *Let S be a minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 6$ and bicanonical map of degree 4: then S is a Burniat surface.*

This result is also somehow surprising, since the Burniat construction is apparently very special, and one would not expect it to include all the possible examples.

Theorem 1.1 also gives us a good understanding of the moduli of the surfaces we are studying: in fact, using natural deformations of $\mathbf{Z}_2 \times \mathbf{Z}_2$-covers, we are able to prove:

Theorem 1.2 *Minimal surfaces S of general type with $p_g(S) = 0$, $K_S^2 = 6$ and bicanonical map of degree 4 form a 4-dimensional irreducible connected component of the moduli space of surfaces of general type.*

The plan of the paper is as follows: in section 2 we explain some facts on irregular double covers of surfaces with $p_g = 0$, which are our main technical tool; in section 3, we recall the construction of Burniat surfaces and we study their deformations; in section 4 we prove that the bicanonical image is smooth: the proof is long and not very enlightening, but, as explained above, this is a key result; in section 5 we collect all the technical facts that we use to prove the main results of section 6.

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Notations and conventions: we work over the complex numbers; all varieties are assumed to be compact and algebraic. We do not distinguish between line bundles and divisors on a smooth variety, and we use both the additive and the multiplicative notation. Linear equivalence is denoted by $\equiv$. All the notation is standard in algebraic geometry; we just recall here the notation for the invariants of a surface S: K_S is the *canonical class*, $p_g(S) = h^0(S, K_S)$ is the *geometric genus* and $q(S) = h^1(S, \mathcal{O}_S)$ is the *irregularity.*

2 Irregular double covers and fibrations

We describe here the key idea of several proofs in this paper.

Let S be a smooth surface, let $D \subset S$ be a smooth curve (possibly empty) and let M be a line bundle on S such that $2M \equiv D$. It is well known that there exist a smooth surface Y and a finite degree 2 map $\pi : Y \to S$ branched over D and such that $\pi_*\mathcal{O}_Y = \mathcal{O}_S \oplus M^{-1}$. We will refer to S as to the *double cover given by the relation* $2M \equiv D$. The invariants of Y are:

$$\begin{aligned} K_Y^2 &= 2(K_S + M)^2 \\ \chi(\mathcal{O}_Y) &= 2\chi(\mathcal{O}_S) + \frac{1}{2}M(K_S + M) \\ p_g(Y) &= p_g(S) + h^0(S, K_S + M) \end{aligned} \tag{2.1}$$

If $p_g(S) = q(S) = 0$, then the existence of a double cover $Y \to S$ with $q(Y) > 0$ forces the existence of a fibration $f : S \to \mathbf{P}^1$ such that the inverse image via π of the general fibre of f is disconnected. This is an old result of De Franchis ([5]), which is explained and generalized in several ways in [4]. However, since these references are perhaps not widely available, we state it here:

Proposition 2.1 *(De Franchis) Let S be a smooth surface such that $p_g(S) = q(S) = 0$ and let $\pi : Y \to S$ be a smooth double cover; if $q(Y) > 0$, then:*

i) the Albanese image of Y is a curve B;

ii) let $\alpha : Y \to B$ be the Albanese fibration: there exist a fibration $g : S \to P^1$ and a degree 2 map $p : B \to P^1$ such that $p \circ \alpha = g \circ \pi$.

Proof: Denote by $\sigma : Y \to Y$ the involution induced by π: σ acts on the Albanese variety of Y as multiplication by -1, since $q(S) = 0$. Given $\eta_1, \eta_2 \in H^0(Y, \Omega^1_Y)$, $\theta = \eta_1 \wedge \eta_2$ is a global 2-form on Y that is invariant for σ, and so it induces an element $\theta' \in H^0(S, K_S)$. Since $p_g(S) = 0$, θ' vanishes identically, and so does θ. So the Albanese image of Y is a curve B. The involution σ acts on Y and on B in a compatible way, and thus the fibration $\alpha : Y \to B$ induces a fibration $g : S \to B/ < \sigma >$. Finally, the quotient curve $B/ < \sigma >$ is isomorphic to $\mathbf{P}^1$, since $q(S) = 0$. $\diamond$

Once constructed such a double cover, sometimes one can reach a contradiction either by showing that the restriction of M to the general fibre of the pencil $g : S \to \mathbf{P}^1$ is nontrivial, and therefore the inverse image via π of a general fibre of f is connected, or by using the following:

Corollary 2.2 *Let S be a minimal surface of general type such that $p_g(S) = q(S) = 0$ and $K^2_S \geq 3$, and let $\pi : Y \to S$ be a smooth double cover: then $K^2_Y \geq 16(q(Y) - 1)$.*

Proof: Since the statement is of course true for $q(Y) \leq 1$, we assume that $q(Y) \geq 2$. By proposition 2.1, the Albanese map of Y is a pencil $\alpha : Y \to B$ and there exists $g : S \to \mathbf{P}^1$ such that $g \circ \pi$ is composed with α. If f is the genus of a smooth fibre of α (and thus of g), then by [2] p. 344 one has: $K^2_Y \geq 8(q(Y) - 1)(f - 1)$. If the inequality in the statement does not hold, then one has $f \leq 2$. Since S is of general type, one must have $f = 2$. On the other hand, by [15] p. 37, S has no genus 2 pencil and so we have a contradiction. $\diamond$

Finally, we also exploit this contruction to show the existence of multiple fibres of fibrations of S, as explained in the following:

Remark 2.3 *Le S be a smooth surface and let $\pi : Y \to S$ be a smooth double cover; let $g : S \to \mathbf{P}^1$ be a fibration such that the general fibre of $g \circ \pi$ is not connected, so that there is a commutative diagram:*

$$\begin{array}{ccc} Y & \stackrel{\pi}{\to} & S \\ {\scriptstyle g'} \downarrow & & \downarrow {\scriptstyle g} \\ B & \stackrel{\bar{\pi}}{\to} & \mathbf{P}^1 \end{array} \qquad (2.2)$$

where B is a smooth curve of genus b and $\bar{\pi}$ is a double cover; if k is the cardinality of the image in $\mathbf{P}^1$ of the branch locus of π, then g has at least $2b+2-k$ fibres that are divisible by 2. In particular, if π is unramified, then g has at least $2b+2$ fibres divisible by 2.

Proof: Let $D \subset \mathbf{P}^1$ be the branch locus of $\bar{\pi}$, let Δ be the branch locus of π and let $D_0 = g(\Delta) \subseteq D$; by the commutativity of the above diagram, $\pi : Y \to S$ is obtained from $\bar{\pi}$ by base change and normalization, and thus $g^*D = \Delta + 2\Delta_0$. Thus $g^*(D - D_0) \subseteq 2\Delta_0$, i.e. the fibres of g over the points of $D - D_0$ are divisible by 2. $\diamond$

3 The Burniat construction

We recall briefly the construction of Burniat surfaces with $K^2 = 6$ (see [12] and [3]), and we describe their bicanonical map and their small deformations.

Let Σ be the blow-up of $\mathbf{P}^2$ at three distinct non collinear points P_1, P_2, P_3: we denote by l the pull-back of a line in $\mathbf{P}^2$, by e_i the exceptional curve corresponding to P_i, by $f_i \equiv l - e_i$ the strict transform of a general line through P_i, $i = 1, 2, 3$, and by e'_i the strict transform of the line joining P_j and P_k, $i \neq j \neq k \neq i$. The subscripts will often be regarded as classes $\bmod 3$. The e'_i's are disjoint -1–curves that also arise as the exceptional curves of a blow-up map $\Sigma \to \mathbf{P}^2$: the two blow-ups are related by a quadratic transformation of $\mathbf{P}^2$ centered at P_1, P_2, P_3. The Picard group of Σ is the free abelian group generated by the classes of l, e_1, e_2, e_3; the anticanonical class $-K_\Sigma \equiv 3l - e_1 - e_2 - e_3 \equiv (f_1 + f_2 + f_3)$ is very ample, and the system $|-K_\Sigma|$ embeds Σ as a smooth surface of degree 6 in $\mathbf{P}^6$, the so-called del Pezzo surface of degree 6.

The Burniat surfaces are $\mathbf{Z}_2 \times \mathbf{Z}_2$-covers of Σ. Denote by $\gamma_1, \gamma_2, \gamma_3$ the nonzero elements of $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$ and by $\chi_i \in \Gamma^*$ the nontrivial character orthogonal to γ_i; by [11], propositions 2.1 and 3.1, to define a smooth Γ-cover $\pi : S \to \Sigma$ one assigns:

i) smooth divisors D_i, $i = 1, 2, 3$, such that $D = D_1 + D_2 + D_3$ is a normal crossing divisor,

ii) line bundles L_1, L_2 satisfying $2L_1 \equiv D_2 + D_3$, $2L_2 \equiv D_1 + D_3$.

The branch locus of π is D: more precisely, D_i is the image of the divisorial part of the fixed locus of γ_i on S. One has $\pi_*\mathcal{O}_S = \mathcal{O}_\Sigma \oplus L_1^{-1} \oplus L_2^{-1} \oplus L_3^{-1}$, where $L_3 = L_1 + L_2 - D_3$ and Γ acts on L_i^{-1} via the character χ_i.

To construct a Burniat surface S with $K_S^2 = 6$, one takes for $i = 1, 2, 3$ two smooth divisors $m_1^i, m_2^i \in |f_i|$ in such a way that no three of the m_j^i's have a common point and sets: $D_1 = e_1 + e_1' + m_1^2 + m_2^2$, $D_2 = e_2 + e_2' + m_1^3 + m_2^3$, $D_3 = e_3 + e_3' + m_1^1 + m_2^1$, $L_1 = 3l - 2e_1 - e_3$, $L_2 = 3l - 2e_2 - e_1$. By the above discussion there exists a smooth Γ-cover $\pi : S \to \Sigma$ corresponding to this choice of data, with $L_3 = 3l - 2e_3 - e_2$. The bicanonical divisor $2K_S = \pi^*(2K_\Sigma + D) = \pi^*(-K_\Sigma)$ is ample, being the pull-back of an ample divisor, and thus S is a minimal surface of general type and $K_S^2 = 4\frac{1}{4}K_\Sigma^2 = 6$. The invariants of S are: $\chi(S) = \chi(\pi_*\mathcal{O}_S) = 1$, $p_g(S) = \sum h^0(\Sigma, K_\Sigma + L_i) = 0$ and thus $q(S) = 0$, since S is of general type.

Proposition 3.1 *Let S be a Burniat surface with $K_S^2 = 6$: then the bicanonical map of S is the composition of the degree 4 cover $\pi : S \to \Sigma$ with the anticanonical embedding of Σ as the smooth Del Pezzo surface of degree 6 in $\mathbf{P}^6$.*

Proof: Since $p_2(S) = 1 + K_S^2 = 7$, the system $\pi^*|-K_\Sigma|$ is complete, and the bicanonical map of S is the composition of π with the anti-canonical embedding of Σ in $\mathbf{P}^6$. $\diamond$

The last part of this section contains some unpublished work of B. Fantechi and the second author on deformations of Burniat surfaces.

Lemma 3.2 *Using the previously introduced notations, one has:*
i) $h^r(\Sigma, T_\Sigma \otimes L_i^{-1}) = 0$, $r \neq 1$, and $h^1(\Sigma, T_\Sigma \otimes L_i^{-1}) = 6$, $i = 1, 2, 3$;
ii) $h^2(\Sigma, T_\Sigma(-\log D_i) \otimes L_i^{-1}) \leq 2$, $i = 1, 2, 3$.

Proof: When no confusion is likely to arise we omit to write the space where cohomology groups are taken. Let $\epsilon : \Sigma \to \mathbf{P}^2$ be the map that blows down e_1, e_2, e_3; pulling back the Euler sequence on $\mathbf{P}^2$ one obtains:

$$0 \to \mathcal{O}_\Sigma \to \mathcal{O}_\Sigma(l)^3 \to \epsilon^* T_{\mathbf{P}^2} \to 0. \tag{3.1}$$

As $-L_i + l \equiv -(f_i + e_{i+1}')$, one has the restriction sequence:

$$0 \to -L_i + l \to \mathcal{O}_\Sigma \to \mathcal{O}_{f_i} \oplus \mathcal{O}_{e_{i+1}'} \to 0.$$

The corresponding long exact sequence gives $h^1(-L_i + l) = 1$, $h^r(-L_i + l) = 0$ for $r \neq 1$. Tensoring 3.1 with L_i^{-1}, we get $h^r(\epsilon^* T_{\mathbf{P}^2} \otimes L_i^{-1}) = 3$ if $r = 1$ and

zero otherwise, since $-L_i$ has no cohomology for $i = 1,2,3$. We have a short exact sequence:

$$0 \to T_\Sigma \to \epsilon^* T_{\mathbf{P}^2} \to \oplus_i \mathcal{O}_{e_i}(-e_i) \to 0. \tag{3.2}$$

Since $(-e_i - L_i)e_i = -1$, $(-e_{i+1} - L_i)e_{i+1} = 1$ and $(-e_{i+2} - L_i)e_{i+2} = 0$, claim i) follows by tensoring 3.2 with L_i^{-1} and considering the corresponding long cohomology sequence. Next we prove ii) for, say, i=1; by Serre duality, $H^2(T_\Sigma(-\log D_1) \otimes L_1^{-1})$ is the dual of $H^0(\Omega^1_\Sigma(\log D_1)(e_2 - e_1)) \subseteq H^0(\Omega^1_\Sigma(\log D_1)(e_2))$. For $i = 2,3$, denote by $\psi_i : \Sigma \to \mathbf{P}^1$ the morphism given by $|f_i|$ and let $\psi = \psi_2 \times \psi_3 : \Sigma \to Q = \mathbf{P}^1 \times \mathbf{P}^1$; ψ contracts e_1 to a point R_1 and e'_1 to a point R'_1, and it is an isomorphism on $\Sigma - (e_1 \cup e'_1)$. Set $M_i = \psi(m_i^2) \in \mathcal{O}_Q(1,0)$, $i = 1,2$, and $N = \psi(e_2) \in \mathcal{O}_Q(0,1)$ and take $\sigma \in H^0(\Omega^1_\Sigma(\log D_1)(e_2))$: then $(\psi^{-1})^*\sigma$ is a section of $\Omega^1_Q(\log M_2 + M_1)(N)$ on $Q - \{R_1, R'_1\}$, and thus $(\psi^{-1})^*\sigma \in H^0(Q, \Omega^1_Q(\log M_2 + M_1)(N))$, since Q is nonsingular. The linear map $(\psi^{-1})^* : H^0(\Omega^1_\Sigma(\log D_1)(e_2)) \to H^0(Q, \Omega^1_Q(\log M_2 + M_1)(N))$ so defined is clearly injective. To finish the proof, we remark $\Omega^1_Q(\log M_2 + M_1)(N) \cong \mathcal{O}_Q(0,1) \oplus \mathcal{O}_Q(0,-1)$, and therefore $h^0(Q, \Omega^1_Q(\log M_2 + M_1)(N)) = 2$. $\diamond$

Proposition 3.3 *Let S be a Burniat surface with $K_S^2 = 6$; then:*
i) the Kuranishi family of S is smooth;
ii) every small deformation of S is also a Burniat surface;

Proof: Again, we omit to write the space where cohomology groups are taken, if no confusion is likely to arise. We denote by $p : \mathcal{X} \to B_0$ the family of natural deformations of the cover $\pi : S \to \Sigma$, defined in section 5 of [11] (for generalizations and a functorial approach to natural deformations see also [6]), we let $B \subset B_0$ be the maximal open subset over which p is smooth, and we let $O \in B$ be the point corresponding to S. Notice that for every $b \in B$ $p^{-1}(b)$ is a Burniat surface, since $H^0(D_i - L_j) = H^0(e_i - e_j) = 0$ for $i \neq j$. The base scheme B is smooth and thus, in order to prove i) and ii) it is enough to show that the characteristic map $\rho : T_{B,0} \to H^1(S, T_S)$ is surjective. Given a vector space V with a Γ-action, we write V^{inv} for the invariant part and $V^{(i)}$ for the subspace on which Γ acts via the character χ_i; Γ acts both on $T_{B,0}$ and $H^1(S, T_S)$ and ρ is equivariant with respect to this action. Thus we have a decomposition $\rho = \rho^{inv} \oplus \rho^1 \oplus \rho^2 \oplus \rho^3$, where $\rho^{inv} : T^{inv}_{B,O} \to H^1(S, T_S)^{inv}$ and

$\rho^i : T^{(i)}_{B,O} \to H^1(S, T_S)^{inv}$. By the definition of natural deformations, we have $T^{inv}_{B,0} = \oplus_i H^0(D_i)$, $T^{(i)}_{B,0} = H^0(D_i - L_{i+1}) \oplus H^0(D_i - L_{i+2})$ and thus $T^{(i)}_{B,0} = 0$ for $i = 1, 2, 3$ by the remark above. By proposition 4.1 of [11], one has $H^1(S, T_S)^{inv} = H^1(\Sigma, T_\Sigma(-\log D))$ and $H^1(S, T_S)^{(i)} = H^1(\Sigma, T_\Sigma(-\log D_i) \otimes L_i^{-1})$, $i = 1, 2, 3$. So we have to show:

i) $H^1(\Sigma, T_\Sigma(-\log D_i) \otimes L_i^{-1}) = 0$ for $i = 1, 2, 3$;

ii) $\rho^{inv} : \oplus_i H^0(\Sigma, D_i) \to H^1(\Sigma, T_\Sigma(-\log D)$ is surjective.

By [11], proposition 5.2, ρ^{inv} is obtained, up to sign, by composing the restriction map $\oplus_i H^0(\Sigma, D_i) \to \oplus_i H^0(\mathcal{O}_{D_i}(D_i))$ with the coboundary map induced by the sequence:

$$0 \to T_\Sigma(-\log D) \to T_\Sigma \to \oplus_i \mathcal{O}_{D_i}(D_i) \to 0 \qquad (3.3)$$

Thus ii) follows from the fact that Σ is rigid and $q(\Sigma) = 0$. Replacing D with D_i in sequence 3.3, tensoring with L_i^{-1} and taking cohomology we get the following sequence:

$$\begin{aligned} 0 \to H^1(T_\Sigma(-\log D_i) \otimes L_i^{-1}) \to H^1(T_\Sigma \otimes L_i^{-1}) \to \\ \to H^1(\mathcal{O}_{D_i}(D_i - L_i)) \to H^2(T_\Sigma(-\log D_i) \otimes L_i^{-1}) \to 0 \end{aligned} \qquad (3.4)$$

Sequence 3.4 is exact on the right by lemma 3.2. The components of D_i are all smooth rational curves and $D_i - L_i = 3e_i - 3e_{i+1}$ has degree -3 on each of them, so that $h^0(\mathcal{O}_{D_i}(D_i - L_i)) = 0$ and $h^1(\mathcal{O}_{D_i}(D_i - L_i)) = 8$. Thus 3.4 is also exact on the left, and ii) follows from lemma 3.2, considering the dimensions of the vector spaces in sequence 3.4. $\diamond$

Theorem 3.4 *Burniat surfaces with $K_S^2 = 6$ form an irreducible open set of dimension 4 of the moduli space of surfaces of general type.*

Proof: As in the proof of proposition 3.3 we consider the family $p : \mathcal{X} \to B$ of smooth natural deformations of S: for every $b \in B$, $p^{-1}(b)$ is a Burniat surface and every Burniat surface occurs as a fibre of p. The image U of B in the moduli space of surfaces of general type is open by proposition 3.3, ii). Denote by $f : B \to \mathbf{P}(H^0(\Sigma, D_1)) \times \mathbf{P}(H^0(\Sigma, D_2)) \times \mathbf{P}(H^0(\Sigma, D_3))$ the restriction to B of the projection map; $f(B)$ is open, 6-dimensional, and the natural map $B \to U$ induces a map $f(B) \to U$. Let $b, b' \in B$ such that there exists an isomorphism $\psi : S \to S'$, where $S = p^{-1}(b)$ and $S' = p^{-1}(b')$:

the covers $\pi : S \to \Sigma$ and $\pi' : S' \to \Sigma$ are given by the bicanonical map, and therefore there exists an automorphism $\bar{\psi}$ of Σ such that $\bar{\psi} \circ \pi = \pi' \circ \psi$. Conversely, given $\bar{\psi} \in Aut(\Sigma)$, then the cover $\pi' : S' \to \Sigma$ given by taking base change of $\pi : S \to \Sigma$ with ψ gives a Burniat surface S' isomorphic to S. So the fibre of $f(B) \to U$ has a map with finite fibres onto $Aut(\Sigma)$ and thus has dimension 2. As a consequence, $\dim U = 4$. $\diamond$

4 The bicanonical image

From now on we will stick to the following

Assumption–Definition 4.1 *We denote by S a smooth minimal surface of general type with invariants $K_S^2 = 6$, $p_g(S) = q(S) = 0$; we denote by $\phi : S \to \Sigma = \phi(S) \subset \mathbf{P}^6$ the bicanonical map, which is a morphism by [13], and assume that* $\deg \phi = 4$. *The surface Σ has degree* 6.

Remark 4.2 *As explained in section 3, Burniat surfaces with $K^2 = 6$ satisfy assumption 4.1.*

We use the notation introduced in section 3.

Theorem 4.3 *Let $\phi : S \to \Sigma$ be as in 4.1: then Σ is the smooth Del Pezzo surface of degree* 6 *in* $\mathbf{P}^6$ *(cf. section 3).*

Proof: The bicanonical image Σ is a linearly normal surface of degree 6; so, by theorem 8 of [10], Σ is the image of $\psi : \hat{\mathbf{P}} \to \mathbf{P}^6$, where $\hat{\mathbf{P}}$ is the blow-up of $\mathbf{P}^2$ at points P_1, P_2, P_3 such that $|-K_{\hat{\mathbf{P}}}|$ has no fixed components, and ψ is given by the system $|-K_{\hat{\mathbf{P}}}|$. Thus the P_i's can be infinitely near, but it is not possible that 2 of them are distinct and both infinitely near to the third one. We denote by l the pull-back on $\hat{\mathbf{P}}$ of a general line in $\mathbf{P}^2$, by e_i the exceptional divisor over P_i, and by l_i a general line through P_i, if P_i is not an infinitely near point; moreover we write L, L_i for the strict transform on S of l, respectively l_i. Σ is smooth iff P_1, P_2, P_3 are distinct and not collinear iff $\hat{P}$ contains no -2–curves; in all the other cases, ψ contracts to rational double points the -2 curves of $\hat{\mathbf{P}}^2$, that are either components of the e_i's or the strict transform of a line containing all the P_i's, if such a line

exists. The proof is a case by case discussion of the possible configurations of the P_i's that give rise to singular Σ's: in each case we consider the pull-back of a general hyperplane section of Σ through one of the singular points, use it to construct an irregular double cover $\pi : Y \to S$ and then obtain a contradiction by means of the techniques of section 2.

Case A: The points P_1, P_2, P_3, not necessarily all distinct, lie on a line m, whose strict transform on $\hat{P}$ is mapped by ψ to a point $x \in \Sigma$.
Considering the pull-back on S of a hyperplane section of Σ through x, one can write: $2K_S = 2L + Z$, where Z is effective with $K_S Z = 0$. In particular, $h^0(S, L) \geq 3$. Write $Z = 2Z' + Z''$, with Z'' reduced; we wish to show $Z'' = 0$. Since $K_S(K_S - L - Z')$ is even, we have $(Z'')^2 = 4(K_S - L - Z')^2 \equiv 0 \pmod 8$. So, if $Z'' \neq 0$, then, being reduced, it contains at least 4 irreducible -2-curves. On the other hand, S contains at most 3 irreducible -2-curves, since $h^{1,1}(S) = 4$ and therefore $Z'' = 0$. If $\pi : Y \to S$ is the unramified double cover given by the relation $2(K_S - L - Z') \equiv 0$, then by the double cover formulas 2.1 we have: $\chi(Y) = 2$, $K_Y^2 = 12$, $p_g(Y) = h^0(S, 2K_S - L - Z') = h^0(S, L + Z') \geq h^0(S, L) = 3$, and therefore $q(Y) \geq 2$. This contradicts corollary 2.2, and thus this case does not occur.

Case B: there is no line containing all the P_i's.
Assume that P_3 is infinitely near to P_2: there are two subcases, according to whether P_2 is infinitely near to P_1 or not.

Case B1: P_2 is not infinitely near to P_1
We write $2K_S = 2L_2 + L_1 + Z$, where Z is effective such that $K_S Z = 0$; one can show that $Z = 2Z'$, with Z' effective, by the argument used in case A. Let $\pi : Y \to S$ be the double cover branched on a general L_1 and given by the relation $2(K_S - L_2 - Z') \equiv L_1$; by the double cover formulas 2.1, one gets $\chi(Y) = 3$, $p_g(Y) = h^0(S, 2K_S - L_2 - Z') = h^0(S, L_1 + L_2 + Z') \geq 4$ and thus $q(Y) \geq 2$. By proposition 2.1, the Albanese image of Y is a curve and there exists a pencil $g : S \to \mathbf{P}^1$ such that $\pi \circ g$ factorizes through the Albanese pencil. Since π is branched on L_1, g must be the map given by $|L_1|$ and thus g has at least 5 fibres divisible by 2, by remark 2.3. Write $D = \phi^*(\psi(e_1))$ and denote by $\bar{D}$ the strict transform of $\psi(e_1)$, so that $D = \bar{D} + Z$ with Z effective and $K_S Z = 0$; one has $D^2 = -4$, $DK_S = \bar{D}K_S = 2$. If R is the ramification divisor of ϕ, by adjunction one has $K_S = R + \phi^* K_\Sigma$, hence $R = 3K_S$; denoting the fibres of g that are divisible by 2 by $2M_i$, $i = 1 \ldots 5$, we have $R \geq \sum_i M_i$.

Assume that $\bar{D}$ is reduced, and thus it has no common component with R: we have $2 = K_S\bar{D} = \frac{1}{3}R\bar{D} \geq \frac{1}{3}\bar{D}\sum_i M_i \geq \frac{5}{6}\bar{D}L_1 = \frac{10}{3}$, and thus we have reached a contradiction. Next we assume that $\bar{D} = 2E$, with E irreducible such that $K_SE = 1$; in this case $L_1E = 2$ and so, for every i, $M_iE = 1$, and the point $M_i \cap E$ is smooth for E and it is a ramification point of the degree 2 map $\phi|_E : E \to \psi(e_1)$. Thus $p_a(E) \geq 2$ by the Hurwitz formula. On the other hand, one has $0 = ZD = 2EZ + Z^2$ and $-4 = D^2 = (2E + Z)^2 = 4E^2 - Z^2$ and thus $E^2 \leq -1$, $p_a(E) \leq 1$. So case B1 does not occur.

Case B2: P_2 is infinitely near to P_1.
As in the previous cases, write $2K_S = 3L_1 + Z$, where Z is effective with $K_SZ = 0$. Since $K_SL_1 = 4$, the index theorem gives either:

a) $L_1^2 = 0$ or,

b) $L_1^2 = 2$.

In addition $8 = 2K_SL_1 = 3L_1^2 + L_1Z$ implies $L_1Z = 8$ in case a) and $L_1Z = 2$ in case b). Taking squares, one gets $24 = 4K_S^2 = 9L_1^2 + 6L_1Z + Z^2$ and thus $Z^2 = -24$ in case a) and $Z^2 = -6$ in case b). The irreducible components of Z are -2–curves and there are at least two of them, since $-Z^2$ is not twice a square. On the other hand, notice that the classes L and $\phi^*(\psi(e_3))$ span a 2-dimensional subspace V in $H^2(\Sigma, \mathbf{Q})$, since they are both effective and satisfy $L^2 = 4$ and $L\phi^*(\psi(e_3)) = 0$. Recalling that $h^2(S, \mathbf{Q}) = 4$ and that the classes of irreducible -2–curves are independent and orthogonal to V, one sees that there are at most 2 such curves on S. So, we denote by θ_1, θ_2 the irreducible -2 curves of S and we write $Z = a_1\theta_1 + a_2\theta_2$, with $a_i > 0$. Observe that $\theta_1\theta_2 \neq 0$, since otherwise we would have integral solutions of $a_1^2 + a_2^2 = 12$, $(= 3$ in case b)). Thus $\theta_1\theta_2 = 1$, since the intersection form is negative definite on the span of θ_1 and θ_2. The equality $Z^2 = -24$ $(= -6$ in case b)) can be rewritten as $(a_1 - a_2)^2 + a_1a_2 = 12$ $(= 3$ in case b)). If we assume $a_1 \geq a_2$, then the only solution is $a_1 = 4$, $a_2 = 2$ in case a) and $a_1 = 2$, $a_2 = 1$ in case b). In addition, $L_1\theta_1 = 2$ in case a), $L_1\theta_1 = 1$ in case b) and $L_1\theta_2 = 0$ in both cases.

Consider now case a) and let $\pi : Y \to S$ be the double cover ramified on a general L_1 and given by the relation $2(K_S - L_1 - 2\theta_1 - \theta_2) \equiv L_1$; we have $\chi(Y) = 3$, $p_g(Y) = h^0(S, 2K_S - L_1 - 2\theta_1 - \theta_2) = h^0(S, 2L_1 + 2\theta_1 + \theta_2) \geq 3$ and thus $q(Y) = 1$. So we argue as in case A, and we see that the pencil $|L_1|$ on S is induced by the Albanese pencil of Y. The curve $\Delta = \pi^*\theta_1$ is not contained in a fibre of the Albanese pencil of Y and it is a smooth rational

curve, since $\theta_1 L_1 = 2$ and L_1 is general. Thus we have a contradiction and case a) is ruled out.

In case b), we consider the double cover $\pi : Y \to S$ branched on $L_1+\theta_2$, L_1 general, given by the relation $2(K_S - L_1 - \theta_1) \equiv \L_1 + \theta_2$; as usual: $\chi(Y) = 3$, $p_g(Y) = h^0(S, 2K_S - L_1 - \theta_1) = h^0(S, 2L_1 + \theta_1 + \theta_2) \geq 3$ and thus $q(Y) \geq 1$. As in the previous cases, the Albanese image of Y is a curve and the Albanese pencil induces a base point free linear pencil $|F|$ on S, that satisfies $L_1 F = 0$; the index theorem applied to L_1, F gives a contradiction, and the proof is complete. $\diamond$

Proposition 4.4 *The canonical divisor K_S of S is ample and ϕ is finite.*

Proof: By theorem 4.3, we have $h^2(\Sigma) = h^2(S) = 4$ and thus the pull-back map $\phi^* : H^2(\Sigma, \mathbf{Q}) \to H^2(S, \mathbf{Q})$, being injective, is an isomorphism preserving the intersection form up to multiplication by 4. If a curve C were contracted by ϕ, then the class of C in $H^2(S, \mathbf{Q})$ would be in the kernel of the intersection form on $H^2(S, \mathbf{Q})$, contradicting Poincaré's duality. So ϕ is finite and, as a consequence, K_S is ample. $\diamond$

5 Divisors, pencils and torsion of S

This section collects all the facts needed in the proof of the main theorem 6.1. By theorem 4.3, if $\phi : S \to \Sigma$ is as in 4.1, then Σ is isomorphic to the blow-up of $\mathbf{P}^2$ at three distinct non collinear points P_1, P_2, P_3 and it is embedded in $\mathbf{P}^6$ by the anticanonical system. We study in great detail the pull-back via ϕ of the exceptional curves and of the free pencils of Σ, and we produce a subgroup $G \simeq \mathbf{Z}_2^3$ of $Pic(S)$ that plays a very important role in the proof of theorem 6.1.

Divisors on Σ are denoted as in section 3.

Lemma 5.1 *Let $\phi : S \to \Sigma$ be as in 4.1 and let $C \subset \Sigma$ be a -1–curve: then either: i) $\phi^* C$ is a smooth rational curve with self-intersection -4; or ii) $\phi^* C = 2E$, where E is an irreducible curve with $E^2 = -1$, $K_S E = 1$.*

Proof: One has: $(\phi^* E)^2 = -4$, $K_S \phi^* E = 2$. So, if $\phi^* E$ is irreducible then it is smooth rational and we are in case i). Assume that $\phi^* E$ is reducible: then $\phi^* E = A + B$, with A, B irreducible and such that $K_S A = K_S B = 1$, K_S

being ample by proposition 4.4. If $A \neq B$, then $AB \geq 0$, $A^2+B^2+2AB=-4$ and so, by parity considerations, either one has $A^2=B^2=-3$, $AB=1$ or, say, $A^2=-3$, $B^2=-1$, $AB=0$. In both cases, the matrix $\begin{pmatrix} A^2 & AB \\ AB & B^2 \end{pmatrix}$ is negative definite, and thus the classes of A and B span a 2-dimensional subspace V of $H^2(S,\mathbf{Q})$. The projection formula: $C\phi_*D = D\phi^*C$, for C and D curves on Σ and S respectively, implies that V and $\phi^*(<E>^{\perp})$ are orthogonal subspaces. By Poincaré's duality, $H^2(\Sigma,\mathbf{Q}) =<E> \oplus^{\perp} <E>^{\perp}$ and thus $H^2(S,\mathbf{Q}) = \phi^* <E> \oplus^{\perp} \phi^*(<E>^{\perp})$, since, as we have already remarked in the proof of proposition 4.4, ϕ^* is an isomorphism preserving the intersection form up to multiplication by 4. Thus $V \subseteq \phi^* <E>$, contradicting the fact that V has dimension 2. So we must have $A=B$ and we are in case ii). $\diamond$

Lemma 5.2 *If S is a surface as in assumption 4.1, then S does not contain 2 smooth disjoint rational curves with self-intersection -4.*

Proof: Let r be the cardinality of a set of smooth disjoint rational curves D on S such that $D^2=-4$; by [9], 2.1, one has the following inequality: $r\frac{25}{12} \leq c_2(S) - \frac{1}{3}K_S^2 = 4$, namely $r \leq 1$. $\diamond$

Proposition 5.3 *Let $\phi : S \to \Sigma$ be as 4.1 and let $e_i, e_i' \subset \Sigma$, $i=1,2,3$, be defined as in section 3: then for $i=1,2,3$ there exist irreducible curves $E_i, E_i' \subset S$ such that $\phi^*e_i = 2E_i$, $\phi^*e_i' = 2E_i'$ and $E_i^2 = (E_i')^2 = -1$, $K_SE_i = K_SE_i' = 1$.*

Proof: By lemmas 5.1 and 5.2, we may assume that there exist irreducible curves E_2, E_3, E_1', E_3' on S such that $E_i^2 = (E_i')^2 = -1$, $K_SE_i = K_SE_i' = 1$ and $\phi^*e_2 = 2E_2$, $\phi^*e_3 = 2E_3$, $\phi^*e_1' = 2E_1'$, $\phi^*e_3' = 2E_3'$ and that ϕ^*e_1, ϕ^*e_2' either are of the same type or they are smooth rational curves. So assume that $\phi^*e_1 = R$ is a smooth rational curve. Writing $F_i = \phi^*f_i$ for $i=1,2,3$, one has: $2K_S \equiv F_1+F_2+F_3 \equiv F_1+R+2E_3'+2E_1'+2E_2 \equiv R+2F_1+2E_1'$. Let $\pi : Y \to S$ be the double cover corresponding to the relation $2(K_S-F_1-E_1') \equiv R$: Y is a smooth surface with invariants $\chi(Y)=2$, $K_Y^2=14$, $p_g(Y) = h^0(S, 2K_S-F_1-E_1') = 3$ (see formulas 2.1). The last equality follows from the fact that ϕ maps F_1 to a conic and E_1' to a line intersecting the conic transversely at one point. Therefore we have $q(Y)=2$ and the result follows from remark 2.2. The proof for E_2' is similar. $\diamond$

Notation 5.4 *Let $\phi : S \to \Sigma$ be as in 4.1. By theorem 4.3, Σ is the blow-up of $\mathbf{P}^2$ at three non colllinear points and we use the notation of section 3 for divisors on Σ; in addition, we write $F_i = \phi^* f_i$ and we denote by $g_i : S \to \mathbf{P}^1$ the morphism given by $|F_i|$, for $i = 1, 2, 3$. Often we use residue classes $(\mathrm{mod}\, 3)$ for the subscripts. For instance, the pencil g_i has two reducible double fibers, that we write as $2E_{i+1} + 2E'_{i+2}$ and $2E_{i+2} + 2E'_{i+1}$. We set: $\eta_i = E_{i+1} + E'_{i+2} - E_{i+2} - E'_{i+1}$, $i = 1, 2, 3$, and $\eta = K_S - (\sum_j E_j + \sum E'_j)$.*

Proposition 5.5 *Let $\phi : S \to \Sigma$ be as in 4.1; let $\eta, \eta_1, \eta_2, \eta_3 \in$ Pic(S) be defined as in 5.4 and let G be the subgroup of* Pic(S) *generated by these elements. Then $G = \{0, \eta_1, \eta_2, \eta_3, \eta, \eta+\eta_1, \eta+\eta_2, \eta+\eta_3\}$, $\eta_1+\eta_2+\eta_3 = 0$, and $G \simeq \mathbf{Z}_2^3$.*

Proof: It is obvious from the definitions that $2\eta = 2\eta_i = 0$ and $\eta_1+\eta_2+\eta_3 = 0$. In addition, $\eta = K_S - \sum_j(E_j + E'_j) \neq 0$ and $\eta + \eta_i = K_S - (E_i + E'_i + 2E'_{i+1} + 2E_{i+2}) \neq 0$, because $p_g(S) = 0$. Finally, $\eta_i \neq 0$, $i = 1, 2, 3$ by [1] lemma (8.3), chap. III. So G consists precisely of the 8 elements listed above. $\diamond$

Lemma 5.6 *If S is as in assumption 4.1, then:*

i) $h^0(S, K_S+\eta) = h^0(S, K_S+\eta_i) = 1$, $h^1(S, K_S+\eta) = h^1(S, K_S+\eta_i) = 0$, $i = 1, 2, 3$;

ii) $h^0(S, K_S + \eta + \eta_i) = 2$, $h^1(S, K_S + \eta + \eta_i) = 1$, $i = 1, 2, 3$;

iii) if $\tau \in Pic(S)$ is such that $2\tau = 0$ and $h^0(S, K_S + \tau) \geq 2$, then $\tau = \eta + \eta_i$ for some $1 \leq i \leq 3$.

Proof: First of all we remark that if $\tau \in Pic(S)$ satisfies $2\tau = 0$, $\tau \neq 0$, then $1 = \chi(K_S + \tau) = h^0(K_S + \tau) - h^1(K_S + \tau)$, and therefore $K_S + \tau$ is effective. Now let $\tau \in Pic(S)$ be such that $2\tau = 0$ and $h^0(S, K_S+\tau) \geq 2$, and write $|K_S + \tau| = Z + |M|$, where Z and $|M|$ are the fixed and the moving part, respectively. The curves $2Z + 2M$ belong to the bicanonical system $|2K_S| = \phi^*| - K_\Sigma|$, and thus $|M| = \phi^*|N|$, where $|N|$ is a linear system of Σ without fixed components such that $-K_\Sigma - 2N$ is effective. The only possibility is $|N| = |f_i|$ for some $i = 1, 2, 3$. In turn, this corresponds to $\tau = \eta + \eta_i$, since $K_S + \eta + \eta_i = F_i + E_i + E'_i$ and $h^0(S, 2(E_i + E'_i)) = 1$. In particular, $h^0(S, K_S + \eta + \eta_i) = 2$. $\diamond$

Lemma 5.7 *If S is a surface as in assumption 4.1, then $|F_1|$, $|F_2|$ and $|F_3|$ are the only irreducible base point free pencils of S.*

Proof: Let D be the cohomology class of a base point free pencil of S: then D lies in the nef cone $NE(S) \subset H^2(S, \mathbf{R})$ and satisfies $D^2 = 0$. Conversely, given $D \in NE(S)$ with $D^2 = 0$ there is at most one irreducible pencil of S whose class is proportional to D.

As we have seen in the proof of corollary. 4.4, $\phi^* : H^2(\Sigma, \mathbf{Q}) \to H^2(S, \mathbf{Q})$ is an isomorphism preserving the intersection form up to multiplication by 4; in addition, integral classes both on S and on Σ are algebraic because $p_g(S) = p_g(\Sigma) = 0$, and therefore $NE(S) = \phi^* NE(\Sigma)$. Now, $NE(\Sigma)$ is spanned by the classes of f_1, f_2, f_3, l, l', where l' is the pull-back of a conic in $\mathbf{P}^2$ through the fundamental points P_1, P_2, P_3, and so D is equal to the class of f_1, f_2 or f_3. $\diamond$

Lemma 5.8 *Let S be as in 4.1 and let $g_i : S \to \mathbf{P}^1$ be as in notation 5.4, $i = 1, 2, 3$; then:*

i) the multiple fibres of g_i are double fibers and their number is ≥ 2 and ≤ 4;

ii) if g_i has 4 double fibres, then E_i and E'_i are smooth elliptic curves.

Proof: We recall that g_i has at least 2 double fibres, namely $2E_{i+1} + 2E'_{i+2}$ and $2E'_{i+1} + 2E_{i+2}$, (see proposition 5.3 and notation 5.4). Let $mD \in |F_i|$, with $m > 1$; since $E_i F_i = E'_i F_i = 2$, one has $m = 2$ and D intersects both E_i and E'_i transversely at smooth points. The irreducible curves E_i and E'_i of arithmetic genus 1 are mapped by ϕ 2–to–1 onto the smooth rational curves e_i and e'_i, and the maps $E_i \to e_i$ and $E'_i \to e'_i$ are ramified at the point DE_i, respectively DE'_i. So, by the Hurwitz formula, there are at most 4 double fibres, and in that case E_i and E'_i are smooth. $\diamond$

Proposition 5.9 *Let S be a surface as in assumption 4.1 and let $F_i \in |F_i|$ be a general curve, $i = 1, 2, 3$; if $i \neq j$, then $F_j|_{F_i} = K_{F_i}$.*

Proof: We show that $F_3|_{F_1} = F_2|_{F_1} = K_{F_1}$. Notice that $2K_S = F_1 + F_2 + F_3 = F_1 + 2E'_3 + 2E_1 + 2E'_2 + 2E_1$, and consider the double cover $\pi : Y \to S$ branched on a smooth F_1 and given by the relation $2(K_S - 2E_1 - E'_3 - E'_1) \equiv F_1$; by the formulas 2.1, the invariants of Y are $\chi(Y) = 3$, $K_Y^2 = 20$, $p_g(Y) =$

$h^0(S, 2K_S - 2E_1 - E_3' - E_2')$. To give a lower bound for $p_g(Y)$ we observe that $|2K_S - 2E_1 - E_3' - E_2'| = |(F_1 + 2E_1) + E_2' + E_3'| = |\phi^* l + E_2' + E_3'| \supseteq \phi^*|l| + E_2' + E_3'$ (see section 3 for the notation) and thus $p_g(Y) = h^0(S, 2K_S - 2E_1 - E_3' - E_2') \geq 3$ and $q(Y) \geq 1$. By proposition 2.1, the Albanese pencil on Y is the pull-back of a pencil $|F|$ on S such that $\pi^* F$ is disconnected for F general. Since π is branched on a curve of $|F_1|$, it follows that $FF_1 = 0$ and therefore $|F| = |F_1|$. In addition, if F_1 is general then $\pi^* F_1$ is the unramified double cover of F_1 given by the relation $2(K_S - 2E_1 - E_3' - E_2')|_{F_1} \equiv 0$; since $\pi^* F_1$ is disconnected, the line bundle $(K_S - 2E_1 - E_3' - E_2')|_{F_1} = (K_S - 2E_1)|_{F_1} = (K_S - F_3)|_{F_1} = (K_S - F_2)|_{F_1}$ is trivial. $\diamond$

Proposition 5.10 *Let S be as in 4.1; for $i = 1, 2, 3$ let $F_i \in |F_i|$ be a general curve and let $G_i = \{\tau \in G : \tau|_{F_i} = 0, \}$: then $G_i = \{\eta_i, \eta + \eta_{i+1}, \eta + \eta_{i+2}\}$.*

Proof: We prove the lemma for G_1. One has $\eta_1 \in G_1$ by definition. Moreover, using lemma 5.9, it is easy to show that $\eta|_{F_1} = \eta_2|_{F_1} = \eta_3|_{F_1} = (E_1 - E_1')|_{F_1}$, so we only need to show $\eta|_{F_1} \neq 0$. Notice that $K_S + F_1 + \eta + \eta_1 = 2F_1 + E_1' + E_1 = 2K_S - E_1 - E_1'$. Therefore $H^0(S, K_S + F_1 + \eta + \eta_1)$ is isomorphic to the kernel of the restriction map $H^0(S, 2K_S) \to H^0(E_1 + E_1', 2K_S|_{E_1 + E_1'})$. Since $|2K_S|$ embeds $E_1 + E_1'$ as a pair of skew lines, it follows that $h^0(S, K_S + F_1 + \eta + \eta_1) = 3$. Next we restrict $K_S + F_1 + \eta + \eta_1$ to F_1 and get: $0 \to H^0(S, K_S + \eta + \eta_1) \to H^0(S, K_S + F_1 + \eta + \eta_1) \to H^0(F_1, K_{F_1}(\eta)) \to H^1(S, K_S + \eta + \eta_1)$. Using lemma 5.6, it follows that $h^0(F_1, K_{F_1}(\eta)) \leq 2$ and so $\eta|_{F_1}$ is nontrivial. $\diamond$

6 The main results

This section is devoted to proving of the following:

Theorem 6.1 *Let S be a smooth minimal surface of general type with invariants $p_g(S) = q(S) = 0$, $K_S^2 = 6$; if the bicanonical map $\phi : S \to \Sigma \subset \mathbf{P}^6$ has degree 4, then S is a Burniat surface.*

and

Theorem 6.2 *Smooth minimal surfaces of general type S with $K_S^2 = 6$, $p_g(S) = 0$ and bicanonical map of degree 4 form a 4-dimensional irreducible connected component of the moduli space of surfaces of general type.*

Proof of theorem 6.2: Let $\mathcal{M}$ be the the moduli space of surfaces of general type with $p_g = 0$ and $K^2 = 6$, and let $\mathcal{Y} \subset \mathcal{M}$ be the subset of surfaces such that the bicanonical map has degree 4: by theorem 6.1 and propositions 3.1 and 3.3, $\mathcal{Y}$ is open in $\mathcal{M}$. In addition (cf. [7]), $\mathcal{Y}$ coincides with the subset of $\mathcal{M}$ consisting of surfaces such that the bicanonical map has degree ≥ 4. In order to show that $\mathcal{Y}$ is also closed, it is enough to prove the following: let B be an irreducible curve and let $f : \mathcal{X} \to B$ be a smooth family, such that for every t the fibre X_t is a minimal surface of general type and the bicanonical map $\phi_t : X_t \to \mathbf{P}^N$ is a generically finite morphism; then there exists m such that $\deg \phi_t \geq m$ for every $t \in B$, with equality holding except for finitely many points $t \in B$. Up to normalizing B and restricting to an open subset, we may assume that there exists $\Phi : \mathcal{X} \to B \times \mathbf{P}^N$ such that $\Phi|_{X_t} = \phi_t$ for every $t \in B$. Denote by $\mathcal{Y}$ the image of $\mathcal{X}$ with the reduced scheme structure: the restriction of the projection $\mathcal{Y} \to B$ is a flat morphism, since B is smooth of dimension 1 and $\mathcal{Y}$ is irreducible. It follows that the fibres Y_t have constant degree d in $\mathbf{P}^N$. For every $t \in B$, let Y'_t the reduced scheme structure underlying Y_t: then one has $4K^2 = \deg \phi_t \deg Y'_t$, and thus $\deg \phi_t \geq m = \frac{4K^2}{d}$, with equality holding iff Y_t is generically reduced. $\diamond$

Proof of theorem 6.1: Since the proof is long, we break it into four steps. We use the notations introduced in sections 3 and 5. In addition, we denote by $\pi_i : Y_i \to S$ the unramified double cover given by $\eta + \eta_i$, for $i = 1, 2, 3$. By the formulas 2.1 and lemma 5.6, we have $p_g(Y_i) = 2$, $q(Y_i) = 1$; we denote by $\alpha_i : Y_i \to B_i$ the Albanese pencil.

Step 1: *Up to a permutation of* $\{1, 2, 3\}$, *the pencil* $g_{i-1} \circ \pi_i : Y_i \to \mathbf{P}^1$ *is composed with* $\alpha_i : Y_i \to B_i$.

By proposition 2.1, the Albanese pencil $\alpha_i : Y_i \to B_i$ arises in the Stein factorization of $g \circ \pi_i$ for some base point free pencil $g : S \to \mathbf{P}^1$. By lemma 5.7, there is $s_i \in \{1, 2, 3\}$ such that $g = g_{s_i}$. Notice that $s_i \neq i$, since by proposition 5.10 the general curve of $\pi_i^*|F_j|$ is connected if and only if $i = j$. To prove the claim, we have to show that $i \mapsto s_i$ is a permutation of $\{1, 2, 3\}$. Assume by contradiction that, say, $s_2 = s_3 = 1$ and denote by $p : Z \to S$ the unramified $\mathbf{Z}_2 \times \mathbf{Z}_2$-cover with data $L_1 = \eta_1$, $L_2 = \eta + \eta_2$, $L_3 = \eta + \eta_3$ (see section 3, or [11] proposition 2.1). One has $q(Z) = \sum_i h^1(S, L_i^{-1}) = 2$ by lemma 5.6; we denote by $\alpha : Z \to A$ the Albanese map. If σ_i is the element of $\mathbf{Z}_2 \times \mathbf{Z}_2$ that acts trivially on L_i^{-1}, then, for $i = 2, 3$, the surface $Z/ < \sigma_i >$ can be naturally identified with Y_i; we denote by $p_i : Z \to Y_i$

the projection map and by $p_{i*} : A \to B_i$ the homomorphism induced by p_i. Notice that $p_{2*} \times p_{3*} : A \to B_2 \times B_3$ is an isogeny, since $H^1(Z, \mathcal{O}_Z) \simeq H^1(S, \eta + \eta_2) \oplus H^1(S, \eta + \eta_3) \simeq p_2^* H^1(Y_2, \mathcal{O}_{Y_2}) \oplus p_3^* H^1(Y_3, \mathcal{O}_{Y_3})$. Since the pencil $g_1 \circ p$ is composed with both $p_{2*} \circ \alpha$ and $p_{3*} \circ \alpha$, the Albanese image of Z is a curve B of genus 2 and $g_1 \circ p = \bar{p} \circ \alpha$, where $\bar{p} : B \to \mathbf{P}^1$ is a $\mathbf{Z}_2 \times \mathbf{Z}_2$-cover. By the Hurwitz formula, $\bar{p}$ is branched exactly over 5 points of $\mathbf{P}^1$, since in a $\mathbf{Z}_2 \times \mathbf{Z}_2$–cover of smooth curves the inverse image of a branch point consists of 2 simple ramification points. Arguing as in the proof of remark 2.3, one sees that the fibres of g_1 over the branch points of $\bar{p}$ are double, but this contradicts lemma 5.8.

Step 2: *The general F_i is hyperelliptic for $i = 1, 2, 3$.*
We show that the general F_1 is hyperelliptic. We have seen that the pencil $g_1 \circ \pi$ is composed with the Albanese map $\alpha_2 : Y_2 \to B_2$ and that $g_3 \circ \pi_2$ also has disconnected fibres. The Stein factorization of $g_3 \circ \pi_2$ is $Y_2 \xrightarrow{g} C \xrightarrow{\psi} \mathbf{P}^1$ where g has connected fibres, C is a smooth curve and $\deg \psi = 2$. Notice that $C \cong \mathbf{P}^1$, since $q(Y_2) = 1$ and g is not the Albanese pencil. Denote by $\tilde{F}_1$ a general fibre of α and by $\tilde{F}_3$ a general fibre of g. From $F_1 F_3 = 4$ it follows that $\tilde{F}_1 \tilde{F}_3 = 2$. So the linear system $|\tilde{F}_3|$ cuts out a g_2^1 on the general $\tilde{F}_1$, and thus the general F_1 is hyperelliptic.

Step 3: *The Galois group Γ of $\phi : S \to \Sigma$ is $\mathbf{Z}_2 \times \mathbf{Z}_2$*
For $i = 1, 2, 3$, denote by γ_i the involution on S that induces the hyperelliptic involution on the general F_i; the γ_i's are regular maps, since S is minimal, and they belong to Γ by proposition 5.9. Consider the involution $\tilde{\gamma_1} : Y_2 \to Y_2$ inducing the hyperelliptic involution on the general $\tilde{F}_1$: by construction $\tilde{\gamma_1}$ maps each $\tilde{F}_3$ to itself, and the restriction of α to $\tilde{F}_3$ identifies $\tilde{F}_3 / < \tilde{\gamma_1} >$ with B_2. Since $\pi_2|_{\tilde{F}_i} : \tilde{F}_i \to \pi_2(\tilde{F}_i) \in |F_i|$ is an isomorphism compatible with the action of $\tilde{\gamma_1}$ and γ_1 for $i = 1, 3$, this implies that $\gamma_1 \neq \gamma_3$. In the same way one shows $\gamma_i \neq \gamma_j$ for $i \neq j$ and thus $\Gamma = \{1, \gamma_1, \gamma_2, \gamma_3\}$.

Step 4: *S is a Burniat surface*
By step 1, for each $i = 1, 2, 3$ the map $g_i \circ \pi_{i+1}$ is composed with the Albanese pencil $\alpha_{i+1} : Y_{i+1} \to B_{i+1}$ and thus, by remark 2.3 and lemma 5.8, g_i has precisely 4 double fibres. The double fibres are $2(E_{i+1} + E'_{i+2})$, $2(E'_{i+1} + E_{i+2})$, and $2M_1^i = \phi^* m_1^i$, $2M_2^i = \phi^* m_2^i$, where $m_1^i, m_2^i \in |f_i|$. If we denote by D the total branch locus of ϕ, then $D \supseteq D_0 = \sum_i (e_i + e'_i + m_1^i + m_2^i)$. By [11] proposition 3.1, D is a normal crossing divisor, since S is smooth, and therefore no three of the m_j^i have a common point. Applying the Hurwitz

formula to a general bicanonical curve yields: $-K_\Sigma D = 18 = -K_\Sigma D_0$ and thus $D = D_0$, since $-K_\Sigma$ is ample. As in section 3, we denote by D_i the image of the divisorial part of the fix locus of γ_i, so that $D = D_1 + D_2 + D_3$. By [11] proposition 3.1, D_i is smooth for every $i = 1, 2, 3$, so there is a permutation $i \mapsto s_i$ of $\{1, 2, 3\}$ such that $D_i \supset m_1^{s_i} + m_2^{s_i}$; in addition, the quotient of a general F_i by γ_i is rational and therefore $D_i f_i = 4$. One concludes that for $i = 1, 2, 3$ $D_i = e_i + e_i' + m_1^{s_i} + m_2^{s_i}$ and $s_i \neq i$. Finally, the quotient of a general F_{i+2} by γ_i is the elliptic curve B_{i+1} (cf. step 3) and thus $D_i f_{i+2} = 2$. So one gets $s_i = i+1$ and S is obtained precisely as explained in section 3. $\diamond$

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